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Dispersive ϕ^4 wave propagation

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Abstract. Whitham's theory of nonlinear water waves is applied to a nonlinear c-number field ($\lambda\phi^4$ model) to investigate the propagation characteristics of the field in the plane-wave modes. A system of first-order partial differential equations is set up for the assumed slow variations of the wave parameters corresponding to each of the different types of Jacobian elliptic function wave modes. By the method of characteristics it is shown that the system is hyperbolic in two of the régimes, indicating that the disturbances in the wave parameters propagate with two different amplitude-dependent velocities. In one of the régimes corresponding to $\lambda < 0$ both of these velocities are tachyonic. These group velocities and the corresponding Riemann invariant forms, which are functions constant along characteristics, are calculated. In two other régimes the system of partial differential equations is elliptic, showing that the wave modes are unstable against formation of inhomogeneities in the wave parameters.

1. Introduction

Non-perturbative techniques for handling problems in quantum field theory (Blokhinstsev and Barbashov 1972) are currently drawing wide attention. One reason for this is that the perturbation theory in terms of the coupling parameters is known to lead often to divergence difficulties such as in the anharmonic oscillator and $\lambda\phi^4$ models (Bender and Wu 1969, Jaffe 1965). Owing to the complexity of the quantum field problems, one often considers simpler analogues for which non-perturbative solutions are possible (Sarkar 1973 for example), and in this way one hopes to gain some feeling for the exact properties of the quantum field. To this end one possibility is to consider the classical counterpart of quantum field problems. It has been observed recently that classical fields in the presence of arbitrary external sources are closely related to the generating functionals of the tree graph approximation of the corresponding quantum field (Boulware and Brown 1968, Duff 1973). There have also been attempts to reduce the solutions of quantum field equations for scalar bosons to those of the corresponding c-number field equation by introducing a suitable integral representation for the quantum field (Rackzka 1974). However there are no known ways of obtaining general solutions, even for the classical nonlinear field equations, since the theory of nonlinear partial differential equations is still in its nascent state at present. Of late various existence studies are appearing in the mathematical literature (for example Segal 1966, Morawetz and Strauss 1972). These are mainly conceptual in nature as they do not provide methods for obtaining the solutions, yet they give some clues as to the nature of the solutions.

In the absence of exact general solutions to nonlinear partial differential equations†, one of the alternatives open at present is to pay attention to particular types of solutions. The one important class of solutions that has been extensively investigated for a variety of nonlinear field equations is the class of localized particle-like solutions both time-dependent and -independent (Anderson and Derrick 1970 for example). However, apart from the conceptual problems involved in such an interpretation, there is the difficulty that solutions of even this limited class are not easily obtainable.

Another important class of particular solutions which one may consider is the wave-like elementary excitations or travelling plane periodic wave solutions of the form

$$\phi(\boldsymbol{x}, t) = \Phi(\boldsymbol{k} . \boldsymbol{x} - \omega t) \tag{1a}$$

with

$$\omega = \omega(k, A_i) \tag{1b}$$

where A_i are integration constants. Various existence theorems for this and other types of periodic solutions to a given equation of the form

$$\phi_{tt}(t, x) + (-)^p D_x^{2p}\phi(t, x) = \lambda F[t, x, \phi, \phi_x, \ldots, D_x^p\phi] \tag{2}$$

(taken for simplicity in $1+1$ space–time dimensions) have been given recently (Hale 1967, Hall 1970). It seems to us that solutions of the kind (1) have a fundamental importance even in the context of quantum field theories. For example, if classical wave modes increase without limit as certain finite values of the amplitude are approached (this happens in the $\lambda\phi^4$ model considered by Mathews and Lakshmanan 1973), such phenomena cannot but have repercussions on the corresponding quantum field theory.

At the classical level itself there have been important new developments in the theory of nonlinear wave propagation following the suggestion of a new method by Whitham (1965a, b). His essential observation is that even though exact general solutions to nonlinear wave equations are out of the question for the present, plane-wave solutions of the form (1) may always be given, at least in principle, for most of the equations of interest, such as the Korteweg–de Vries equation (Korteweg and de Vries 1895). Whitham shows that for a more general category of solutions, consisting of those which can be approximated by plane waves *locally* (so that they can be represented by (1) with k, A_i replaced by the slowly varying functions $k(x, t)$, $A_i(x, t)$ which vary little over several wavelengths), the temporal behaviour can be deduced from consideration of a suitable set of conservation equations. The theory provides a natural extension of the group velocity concept to nonlinear problems. There are then possibilities of studying along these lines certain typical nonlinear field equations of current interest in quantum field theory.

The purpose of the present paper is to investigate the propagation characteristics of inhomogeneities of plane-wave modes of the classical $\lambda\phi^4$ model, characterized by the equation of motion

$$\partial^2\phi(\boldsymbol{x}, t)/\partial t^2 - \nabla^2\phi + m^2\phi + \lambda\phi^3 = 0. \tag{3}$$

† It might be of interest to note Einstein's view regarding the role of classical nonlinear field equations in describing atomic and quantum structure of reality. He observes that nobody really knows anything about this at present, as there is no way of obtaining exact general solutions to classical nonlinear field equations (Einstein 1966).

Various studies in recent mathematical literature indicate the existence of a classical nonlinear scattering operator S, and show that the solutions are asymptotic in the energy norm to solutions of the linear equation, ie $\lambda = 0$ as $t \to \pm\infty$ (Morawetz and Strauss 1973) with m^2, $\lambda > 0$. However, as far as analytic solutions are concerned, it appears that the only type of solution we can obtain explicitly at present is a plane-wave solution of the form (1). A systematic study and classification of non-singular wave solutions of the type (1) to equation (3) was made recently (Mathews and Lakshmanan 1973, hereafter referred to as I). Complete knowledge of the plane-wave solutions enables us to apply Whitham's theory to our system and obtain information about the propagation of wave patterns described by more general solutions of the equation, such as groups of waves wherein the propagation constant k and the amplitude A vary slowly with position and time. In an earlier paper (Lakshmanan 1974, hereafter referred to as II) we have made such an analysis for a model with non-polynomial interaction (Mathews and Lakshmanan 1974), where an outline of Whitham's theory has also been given.

Starting from the appropriate conservation equations we obtain the system of partial differential equations satisfied by the wave parameters k and A of the $\lambda\phi^4$ model (§ 2). It turns out that this system of equations may be either hyperbolic or elliptic depending on the particular régime under consideration. In fact two of the régimes for $\lambda < 0$ lead to elliptic systems, indicating unstable propagation of the wave modes with respect to formation of inhomogeneities in the wave parameters. In the cases where the equations are hyperbolic we have proceeded to find the 'group velocities' of propagation of wave patterns and also obtained Riemann invariant forms (which are functions constant along characteristics). Very recently Hayes (1973) has suggested methods to investigate elliptic cases. However we have not considered this problem in this paper.

The main results of the paper are as follows: we find in § 3 that for $\lambda > 0$ the assumed slow variations in the wave parameters of modes with $\omega^2 - k^2 > 0$ propagate with two different amplitude-dependent group velocities, as in the derivatively coupled non-polynomial case considered earlier by us (II). In § 4 we show that when $\lambda < 0$ there is no propagation of the inhomogeneities of the wave parameters of the *sn* wave mode with $\omega^2 - k^2 > 0$ and of the *dn* wave mode which is 'tachyonic' ($\omega^2 - k^2 < 0$). However, for the 'tachyonic' type wave mode (which also exists when $\lambda < 0$ and must have an amplitude exceeding the upper limit $(2m^2/|\lambda|)^{1/2}$ of the *dn* type waves) inhomogeneities in the wave parameters propagate with two different tachyonic group velocities.

2. Locally plane-wave solutions; partial differential equations for the wave parameters

The appropriate averaged conservation equations (along the lines of Whitham's theory discussed in II) that one uses here for the $\lambda\phi^4$ case are

$$\frac{\partial \bar{\mathscr{H}}}{\partial t} + \frac{\partial \bar{\mathscr{P}}}{\partial x} = 0 \tag{4a}$$

and

$$\frac{\partial \bar{\mathscr{P}}}{\partial t} + \frac{\partial \bar{\mathscr{T}}}{\partial x} = 0. \tag{4b}$$

The quantities $\mathscr{H}$, $\bar{\mathscr{P}}$ and $\mathscr{T}$ to be substituted in the averaged conservation equations (4) are

$$\bar{\mathscr{H}} = \frac{1}{l}\int_0^l \frac{1}{2}\left[\left(\frac{\partial\phi}{\partial t}\right)^2 + \left(\frac{\partial\phi}{\partial x}\right)^2 + m^2\phi^2 + \frac{\lambda}{2}\phi^4\right] \mathrm{d}x, \tag{5a}$$

$$\bar{\mathscr{P}} = -\frac{1}{l}\int_0^l \frac{\partial\phi}{\partial x}\frac{\partial\phi}{\partial t}\,\mathrm{d}x \tag{5b}$$

and

$$\bar{\mathscr{T}} = \frac{1}{l}\int_0^l \frac{1}{2}\left[\left(\frac{\partial\phi}{\partial t}\right)^2 + \left(\frac{\partial\phi}{\partial x}\right)^2 - m^2\phi^2 - \frac{\lambda}{2}\phi^4\right] \mathrm{d}x. \tag{5c}$$

Here l is the wavelength and ϕ is to be substituted in terms of the various wave solutions $\phi(x, t) = f(kx - \omega t)$ (in $1+1$ space–time dimensions) that we have discussed in I (see table I). One may observe that equations (4) are nothing but the energy–momentum conservation equation restricted to $1+1$ dimensions (see II). In I we have distinguished four cases of interest:

(i) Case 1: $\lambda > 0$, $\omega^2 - k^2 > 0$, *cn* type solution
(ii) Case 2: $\lambda < 0$, $\omega^2 - k^2 > 0$, *sn* type solution

$(0 \leqslant |A| \leqslant (m^2/|\lambda|)^{1/2})$

(iii) Case 3: $\lambda < 0$, $\omega^2 - k^2 < 0$, *dn* type solution

$((m^2/|\lambda|)^{1/2} \leqslant |A| \leqslant (2m^2/|\lambda|)^{1/2})$

(iv) Case 4: $\lambda < 0$, $\omega^2 - k^2 < 0$, *cn* type solution

$((2m^2/|\lambda|)^{1/2} \leqslant |A| \leqslant \infty)$.

By inspection of equations (17) and (18) of I one can see that the averaged values of $\bar{\mathscr{H}}_i$ and $\bar{\mathscr{P}}_i$ are

$$\bar{\mathscr{H}}_i = \alpha + \tfrac{1}{3}\beta_i A^2 k^2 \qquad \bar{\mathscr{P}}_i = \tfrac{1}{3}\beta_i A^2 \omega k \qquad i = 1, 2, 3, 4, \tag{6}$$

where α and β are given by

$$\alpha = \tfrac{1}{2}A^2 m^2 (1 + \lambda A^2/2m^2) \tag{7a}$$

$$\beta_1 = \beta_4 = 1 + \left(\frac{2m^2}{\lambda A^2}\right)\left(1 - \frac{E}{K}\right)$$

$$\beta_2 = 2 + \left(\frac{2m^2}{\lambda A^2}\right)\left(1 - \frac{E}{K}\right) \tag{7b}$$

$$\beta_3 = 2\left(1 + \frac{m^2}{\lambda A^2}\right) + \left(\frac{2m^2}{\lambda A^2}\right)\left(1 - \frac{E}{K}\right).$$

Here K and E are the complete elliptic integrals of the first and second kind respectively. One may also evaluate $\bar{\mathscr{T}}_i$ in a similar manner. We find that

$$\bar{\mathscr{T}}_i = -\alpha + \tfrac{1}{3}\beta_i A^2 \omega^2. \tag{8}$$

Substituting these expressions in the averaged conservation equations (4), we obtain the set of approximate conservation equations for each of the four cases in the form:

$$L_1 \equiv a_i k_t + b_i k_x + c_i A_t + d_i A_x = 0 \tag{9a}$$

$$L_2 \equiv b_i k_t + a_i k_x + d_i A_t + e_i A_x = 0 \tag{9b}$$

where the coefficients a_i, b_i, c_i, d_i, e_i ($i = 1, 2, 3, 4$) are functions of the parameters k and A. Their actual forms are given at the appropriate places below as we analyse the four different cases separately.

3. Case 1: $\lambda > 0$ and $\omega^2 - k^2 > 0$

This case corresponds to the *cn* solutions with modulus $\eta = [\lambda A^2/2(m^2+\lambda A^2)]^{1/2}$ and characterized by the dispersion relation $\omega^2 - k^2 = m^2 + \lambda A^2$. The various coefficients a_1, b_1, c_1, d_1, e_1 in equation (9) are obtained by substituting equations (6) and (8) with (7) in the conservation equations. They are

$$a_1 = 2k, \tag{10a}$$

$$b_1 = \omega + k^2/\omega, \tag{10b}$$

$$c_1 = [3(m^2+\lambda A^2) + 2k^2(1-\Omega_1)]\chi_1^{-1}\lambda A, \tag{10c}$$

$$d_1 = [2\omega k(1-\Omega_1)\chi_1^{-1} + k/\omega]\lambda A, \tag{10d}$$

$$e_1 = [-3(m^2+\lambda A^2) + 2\chi_1 + 2\omega^2(1-\Omega_1)]\chi_1^{-1}\lambda A \tag{10e}$$

where

$$\Omega_1 = \frac{m^4}{\lambda A^2}\frac{1}{(m^2+\lambda A^2)}\left[\frac{2E}{K} - \left(1 + \frac{2(m^2+\lambda A^2)}{(2m^2+\lambda A^2)}\frac{E^2}{K^2}\right)\right] \tag{10f}$$

and

$$\chi_1 = 2m^2(1-E/K) + \lambda A^2. \tag{10g}$$

In deriving the coefficients of the system of partial differential equations (9) we have used the following results on the elliptic functions:

$$\frac{\mathrm{d}E}{\mathrm{d}\eta} = \frac{E-K}{\eta}, \qquad \frac{\mathrm{d}K}{\mathrm{d}\eta} = \frac{(E-\eta'^2 K)}{\eta\eta'^2}, \tag{11a}$$

so

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\frac{E}{K}\right) = \frac{1}{\eta}\left[\frac{2E}{K} - \left(1 + \frac{1}{\eta'^2}\frac{E^2}{K^2}\right)\right]. \tag{11b}$$

Before proceeding to the determination of the nature of the partial differential equations (9) we verify that the approximate conservation equations (9) also imply the conservation of waves. We first multiply equation (9*a*) by k/ω and subtract from (9*b*) to obtain

$$\left(b_1 - a_1\frac{k}{\omega}\right)k_t + \left(a_1 - b_1\frac{k}{\omega}\right)k_x + \left(d_1 - e_1\frac{k}{\omega}\right)A_t + \left(e_1 - d_1\frac{k}{\omega}\right)A_x = 0. \tag{12}$$

On substituting the actual values of coefficients $a_1, \ldots, e_1$, (9) becomes

$$k_t + \frac{k}{\omega}k_x + \{2(1-\Omega_1) + [\chi_1 - 3(m^2+\lambda A^2)](m^2+\lambda A^2)^{-1}\}\lambda k A \chi_1^{-1} A_t$$
$$+ \left([\chi_1 - 3(m^2+\lambda A^2)](m^2+\lambda A^2)^{-1} + 2(1-\Omega_1) + \frac{\lambda}{\omega^2}\chi_1\right)\lambda\omega A \chi_1^{-1} A_x = 0 \tag{13}$$

which may be simplified to

$$k_t + \frac{k}{\omega}k_x + \frac{k}{A}\frac{(1-2\eta^2)}{\eta^2}\left[\eta^2 - \left(1-\frac{E}{K}\right)\right]A_t$$
$$+ \frac{1}{A}\left\{\frac{\omega}{\eta^2}(1-2\eta^2)\left[\eta^2 - \left(1-\frac{E}{K}\right) + \frac{\lambda A^2}{\omega}\right]\right\}A_x = 0. \tag{14}$$

Using the second of the formulae given in (11*a*) one may verify that equation (14) is exactly equivalent to

$$\partial_t\left(\frac{k}{4K}\right) + \partial_x\left(\frac{\omega}{4K}\right) = 0. \tag{15}$$

Since $k/4K$ is the density of waves (or wavenumber) and $\omega/4K$ the flux of the waves (or frequency), we see that equation (15) verifies the conservation of the number of waves.

We now turn to the investigation of the nature of system (9) for case 1. The nature of the partial differential equations is determined by the nature of the roots of the characteristic equation (Courant and Hilbert 1962, p 172, Jeffrey and Taniuti 1964):

$$Q \equiv \begin{vmatrix} b_1 - \tau a_1 & d_1 - \tau c_1 \\ a_1 - \tau b_1 & e_1 - \tau d_1 \end{vmatrix} = 0. \tag{16}$$

On actual substitution of the relevant coefficients from equation (10) this becomes, after some simplification,

$$\{(m^2+\lambda A^2)[3(2k^2+m^2+\lambda A^2) - 2k^2(1-\Omega_1)] - 2k^2\chi_1\}\tau^2 - 4k\omega[(m^2+\lambda A^2)(2+\Omega_1) - \chi_1]\tau$$
$$+ 3(\omega^2+k^2)(m^2+\lambda A^2) - 2\omega^2\chi_1 - 2\omega^2(m^2+\lambda A^2)(1-\Omega_1) = 0. \tag{17}$$

The solution of this quadratic equation gives the characteristic roots as

$$\tau = \frac{2k\omega[(m^2+\lambda A^2)(2+\Omega_1) - \chi_1] \pm (m^2+\lambda A^2)^{3/2}[2\chi_1 - (m^2+\lambda A^2)(1+2\Omega_1)]^{1/2}\sqrt{3}}{(m^2+\lambda A^2)[2k^2(2+\Omega_1) + 3(m^2+\lambda A^2)] - 2k^2\chi_1}. \tag{18}$$

These roots will be real or complex according as $2\chi_1 - (m^2+\lambda A^2)(1+2\Omega_1)$ is positive or negative. Examination of (10*f*) and (10*g*) shows that in the limit $\lambda A^2 \to 0(\eta \to 0)$, $\Omega_1 \to -\frac{1}{2}$ and $\chi_1 \to 0$, while if $\lambda A^2 \to \infty$, $\Omega_1 \to 0(\eta \to 1/\sqrt{2})$ and $\chi_1 \to \lambda A^2 (= \infty)$. Thus the quantity $2\chi_1 - (m^2+\lambda A^2)(1+2\Omega_1)$ tends monotonically from zero to $\lambda A^2 (= \infty)$ as η^2 varies from zero to a half, showing that it is always positive definite. Therefore system (9) for case 1 is hyperbolic in nature, and equation (18) gives the corresponding propagation velocities of inhomogeneities in the parameters k and A. We also note that in the limit $\lambda \to 0$,

$$\tau \to k/\omega \tag{19}$$

which is the usual group velocity of linear dispersive waves. When $m^2 = 0$ or $\lambda \to \infty$,

$$\tau_{m^2=0} \to (2\omega k \pm \lambda A^2\sqrt{3})/(2k^2+3\lambda A^2). \tag{20}$$

In all three equations (18)–(20), $|\tau|$ is always less than unity. We also note that the two separate propagation velocity expressions (18) become the same in the $\lambda = 0$ limit, while in the limit $m^2 = 0$ there is again propagation with two different amplitude-dependent velocities. This phenomenon is absent in the corresponding limit of the model with non-polynomial interaction (II) because in that case there is no dispersion in the $m^2 = 0$ limit.

To obtain the characteristic form of (9) for case 1 (wherein the differentiation is explicitly along the characteristic directions), we proceed as below†. We consider first the non-trivial solutions of the equation

$$\begin{pmatrix} b_1-\tau a_1 & a_1-\tau b_1 \\ d_1-\tau c_1 & e_1-\tau d_1 \end{pmatrix}\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = 0 \tag{21}$$

where τ is taken to have either of the values of (18). Evidently

$$\frac{r_1}{r_2} = -\frac{a_1-\tau b_1}{b_1-\tau a_1} = -\frac{e_1-\tau d_1}{d_1-\tau c_1}. \tag{22}$$

With either of these values for r_1/r_2 the linear combination $r_1L_1+r_2L_2$ of equations (9) involves differentiation of k and A in the characteristic directions only. Accordingly we have the system of equations (with $\tau = \tau_i$, $i = 1, 2$):

$$k_t+\frac{(r_1/r_2)b+a}{(r_1/r_2)a+b}k_x+\frac{(r_1/r_2)c+d}{(r_1/r_2)a+b}A_t+\frac{(r_1/r_2)d+e}{(r_1/r_2)a+b}A_x = 0.$$

On substituting (22) for r_1/r_2 this becomes

$$(k_t+\tau k_x)+\frac{(b_1d_1-c_1a_1)+\tau(b_1c_1-a_1d_1)}{b_1^2-a_1^2}(A_t+\tau A_x) = 0, \tag{23}$$

which on simplification with the aid of equation (10) becomes

$$\mathrm{d}k+\left(-\frac{\lambda kA}{m^2+\lambda A^2}\mp\frac{\omega\chi_1^{-1}\sqrt{3}}{(m^2+\lambda A^2)^{1/2}}[2\chi_1-(m^2+\lambda A^2)(1+2\Omega_1)]^{1/2}\right)\mathrm{d}A = 0 \tag{24a}$$

on the characteristic curves

$$C_\pm : v_{gi} \equiv \tau_i = \mathrm{d}x/\mathrm{d}t \qquad i = 1, 2 \tag{24b}$$

where τ_i are given by equation (18). To obtain the Riemann invariant forms we define $U = k/\omega$ so that

$$k_t-\frac{\lambda Ak}{m^2+\lambda A^2}A_t = \frac{}{m^2+\lambda A^2}U_t, \qquad k_x-\frac{\lambda Ak}{m^2+\lambda A^2}A_x = \frac{\omega^3}{m^2+\lambda A^2}U_x. \tag{25}$$

† We follow in part the procedure of Courant and Hilbert (1962, p 429) on the isentropic flow of compressible fluids.

Then the characteristic form (24) may be re-expressed as a quadrature in the form

$$\frac{\mathrm{d}U}{1-U^2} \pm \frac{\lambda\chi_1^{-1}A\sqrt{3}}{(m^2+\lambda A^2)^{1/2}}[2\chi_1^{-1}-(m^2+\lambda A^2)(1+2\chi_1)]^{1/2}\,\mathrm{d}A = 0, \tag{26a}$$

the characteristic curves being given by

$$C_{\pm}:\tau_i = \frac{2U[(m^2+\lambda A^2)(2+\Omega_1)-\chi_1] \pm(1-U^2)(m^2+\lambda A^2)^{1/2}[2\chi_1-(m^2+\lambda A^2)(1+2\Omega_1)]^{1/2}\sqrt{3}}{(m^2+\lambda A^2)[2U^2(2+\Omega_1)+3(1-U^2)]-2U^2\chi_1}. \tag{26b}$$

The Riemannian invariant forms are to be obtained by integration of (26*a*). In particular, if $m^2 = 0$ the characteristic curves are

$$C_{\pm}:\tau_i = \frac{2U\pm(1-U^2)\sqrt{3}}{2+3(1-U^2)} \tag{27a}$$

and (26*a*) reduces to

$$\frac{\mathrm{d}U}{1-U^2}\pm\frac{\sqrt{3}}{A}\,\mathrm{d}A = 0. \tag{27b}$$

Then the corresponding Riemann invariants are obtained by integration of (27*a*):

$$\ln\left[\left(\frac{1+U}{1-U}\right)A^{\pm\sqrt{3}}\right] = \text{constant} \tag{28}$$

corresponding to the + and − signs respectively. If we call

$$\ln\left[\left(\frac{1+U}{1-U}\right)A^{\sqrt{3}}\right] = r \qquad \text{and} \qquad \ln\left[\left(\frac{1+U}{1-U}\right)A^{-\sqrt{3}}\right] = s \tag{29}$$

then we have

$$U = \frac{\mathrm{e}^{(r+s)/2}-1}{\mathrm{e}^{(r+s)/2}+1}, \qquad A = \mathrm{e}^{(r-s)/2\sqrt{3}}. \tag{30}$$

Expressions (29) may then be used to solve the initial value problem. For this purpose we introduce r and s as parameters instead of U and A. In terms of r and s the characteristic equations $\mathrm{d}x/\mathrm{d}t = \tau_i$ may be given in the form†:

$$x_s = \tau_1 t_s, \qquad x_r = \tau_2 t_r. \tag{31}$$

This may then be reduced (Courant and Friedrichs 1948) to a single second-order linear partial differential equation of the form:

$$(\tau_1-\tau_2)t_{rs}+(\tau_1)_r t_s-(\tau_2)_s t_r = 0. \tag{32a}$$

† This transformation is valid as long as the Jacobian $\partial(x,t)/\partial(r,s) \equiv x_r t_s - x_s t_r$ is different from zero, which we have assumed here. For details see Courant and Hilbert (1962, p 492).

On simplification this becomes

$$t_{rs}+\left(\frac{(2-\sqrt{3})\,e^{r+s}+2\,e^{(r+s)/2}+(2+\sqrt{3})}{e^{r+s}+4\,e^{(r+s)/2}+1}\right)t_s$$

$$-\left(\frac{(2+\sqrt{3})\,e^{r+s}+2\,e^{(r+s)/2}+(2-\sqrt{3})}{e^{r+s}+4\,e^{(r+s)/2}+1}\right)t_r = 0. \tag{32b}$$

The substitution of the solutions of this equation for t into equation (31) gives x in terms of r and s. Inverting the solutions we will obtain r and s (which in turn means k and A) in terms of x and t. Thus the initial value problem may be solved in principle by solving this linear differential equation.

4. The case $\lambda < 0$

4.1. *Normal solutions* $(\omega^2-k^2>0)$

The solution pertaining to this case is $\phi = A\,sn(\boldsymbol{k}\,.\,\boldsymbol{x}-\omega t+\theta)$ with $\omega^2-k^2 = m^2-|\lambda|A^2/2$ and $\eta^2 = |\lambda|A^2/(2m^2-|\lambda|A^2)$. The corresponding coefficients a, b, c, d, e in the partial differential equations (9) are

$$a_2 = 2k, \qquad b_2 = \omega+k^2/\omega, \tag{33a}$$

$$c_2 = [\tfrac{3}{2}(m^2-|\lambda|A^2)+2k^2(1+\Omega_2)]\lambda A\chi_2^{-1}, \tag{33b}$$

$$d_2 = [2\omega k(1+\Omega_2)\chi_2^{-1}-\tfrac{1}{2}(k/\omega)]\lambda A, \tag{33c}$$

$$e_2 = [\tfrac{3}{2}(|\lambda|A^2-m^2)-\chi_2+2\omega^2(1+\Omega_2)]A\chi_2^{-1} \tag{33d}$$

where

$$\Omega_2 = \frac{m^4}{|\lambda|A^2}\,\frac{1}{(2m^2-|\lambda|A^2)}\left[\frac{2E}{K}-\left(1+\frac{(2m^2-|\lambda|A^2)}{2m^2}\,\frac{E^2}{K^2}\right)\right] \tag{33e}$$

and

$$\chi_2 = |\lambda|A^2-m^2\left(1-\frac{E}{K}\right). \tag{33f}$$

As before, we consider the roots τ_i of the characteristic equation

$$\begin{vmatrix} b_2-\tau a_2 & d_2-\tau c_2 \\ a_2-\tau b_2 & e_2-\tau d_2 \end{vmatrix} = 0, \tag{34}$$

namely

$$\tau_i = \frac{-a_2(e_2-c_2)\pm[a_2^2(e_2-c_2)^2-4(a_2d_2-b_2c_2)(b_2e_2-a_2d_2)]^{1/2}}{2(a_2d_2-b_2c_2)}. \tag{35}$$

The quantity under the square root sign in the numerator turns out to be negative:

$$\begin{aligned} &a_2^2(e_2-c_2)^2-4(a_2d_2-b_2c_2)(b_2e_2-a_2d_2)\\ &\quad = \tfrac{3}{2}(\omega^2-k^2)^2(m^2-|\lambda|A^2)[-\tfrac{3}{2}(m^2-|\lambda|A^2)-\chi_2+(2m^2-|\lambda|A^2)(1+\Omega_2)]/\omega^2\chi_2^2\\ &\quad \leqslant 0. \end{aligned} \tag{36}$$

This is because the function Ω_2 of equation (33*e*) varies from† $-\frac{1}{4}$ to $-\infty$ as $|\lambda|A^2$ varies from zero to its maximum allowed value m^2. A study of (36) shows that as long as $(2m^2-|\lambda|A^2)(1+\Omega_2)$ is positive, the other two terms inside the square bracket of (36) dominate (and $\chi_2 \geqslant 0$ in this range). Thus the above expression is always negative for the whole range $0 \leqslant |A| \leqslant (m^2/|\lambda|)^{1/2}$. Hence the roots (35) are complex, making the corresponding system of partial differential equations elliptic. Therefore there is no propagation of the inhomogeneities in this case.

4.2. *Tachyonic wave modes* ($\omega^2 - k^2 < 0$)

In this region we have two types of solution: (i) case 3: *dn* type wave modes with amplitudes in the range $(m^2/|\lambda|)^{1/2} \leqslant |A|(2m^2/|\lambda|)^{1/2}$; and (ii) case 4: *cn* type waves having amplitudes $A > (2m^2/|\lambda|)^{1/2}$.

4.2.1. Case 3: dn wave modes. The dispersion relation for this case is $\omega^2 - k^2 = -|\lambda|A^2/2$ and the modulus of the *dn* function is $\eta = [2(|\lambda|A^2 - m^2)/|\lambda|A^2]^{1/2}$. The corresponding coefficients $a_3, \ldots, e_3$ of the partial differential equations (9) are given by

$$a_3 = 2k, \qquad b_3 = \omega + k^2/\omega, \tag{37a}$$

$$c_3 = [\tfrac{3}{2}(m^2 - |\lambda|A^2) + 2k^2(1+\Omega_3)]|\lambda|A^2\chi_3^{-1}, \tag{37b}$$

$$d_3 = [-k/\omega + 2k\omega(1+\Omega_3)\chi_3^{-1}]|\lambda|A, \tag{37c}$$

$$e_3 = [\tfrac{3}{2}(|\lambda|A^2 - m^2) - \chi_3 + 2\omega^2(1+\Omega_3)]|\lambda|A\chi_3^{-1} \tag{37d}$$

where

$$\Omega_3 = \frac{m^4}{|\lambda|A^2}\,\frac{1}{2(|\lambda|A^2 - m^2)}\left(\frac{2E}{K} - \frac{|\lambda|A^2}{(2m^2 - |\lambda|A^2)}\,\frac{E^2}{K^2}\right) \tag{37e}$$

and

$$\chi_3 = \left[|\lambda|A^2 - m^2\left(2 - \frac{E}{K}\right)\right]. \tag{37f}$$

The roots of the characteristic equation are now given by expression (35) with $a_2, b_2,$

† To determine the behaviour of Ω as $|\lambda|A^2 \to m^2$ (or $\eta^2 \to 1$) in the expression

$$\Omega_2 = \frac{m^4}{|\lambda|A^2}\,\frac{1}{(2m^2 - |\lambda|A^2)}\left[\frac{2E}{K} - \left(1 + \frac{1}{\eta'^2}\,\frac{E^2}{K^2}\right)\right],$$

we should determine the behaviour of $\eta' K(\eta)$ (where $\eta'^2 = 1 - \eta^2$) in the limit $\eta^2 \to 1$. (We note that $E \to 1$ as $\eta \to 1$). This may be done with the aid of the following expansion for the complete elliptic integral K (Gradshteyn and Ryzhik 1965):

$$K = \ln\left(\frac{4}{\eta'}\right) + (\tfrac{1}{2})^2\left(\ln\frac{4}{\eta'} - \frac{2}{1\cdot 2}\right)\eta'^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2\left(\ln\frac{4}{\eta'} - \frac{2}{1\cdot 2} - \frac{2}{3\cdot 4}\right)\eta'^4 + \ldots.$$

So

$$\lim_{\eta \to 1} \eta' K(\eta) \to \eta'(\ln 4 - \ln \eta') = \lim_{\eta' \to 0} -\eta' \ln \eta' = 0.$$

Therefore $\Omega_2 \to -\infty$.

c_2, d_2, e_2 changed to a_3, b_3, c_3, d_3, e_3 respectively. We find that the term

$$a_3^2(e_3-c_3)^2-4(a_3d_3-b_3c_3)(b_3e_3-a_3d_3)$$
$$=\tfrac{3}{2}(\omega^2-k^2)^2(|\lambda|A^2-m^2)[-\tfrac{3}{2}(|\lambda|A^2-m^2)+\chi_3+|\lambda|A^2(1+\Omega_3)]/\omega^2\chi_3^2 \leqslant 0 \quad (38)$$

so that τ_i are complex. Thus the system of partial differential equations (9) is again elliptic and there is no stable propagation of inhomogeneities in this case.

4.2.2. cn wave modes (tachyonic). The dispersion formula here (case 4) is

$$\omega^2-k^2 = -(|\lambda|A^2-m^2) \qquad (|\lambda|A^2>2m^2)$$

and the modulus of the elliptic function is $\eta = [|\lambda|A^2/2(|\lambda|A^2-m^2)]^{1/2}$. The coefficients in the corresponding partial differential equations (9) are of the same form as those of case 1 considered in § 2; but now the modulus η will vary from $\eta = 1$ at $|\lambda|A^2 = 2m^2$ to $\eta = 1/\sqrt{2}$ as $|\lambda|A^2 \to \infty$. The details are then similar to § 3 and we may write down the characteristic roots as

$$\tau = \frac{2k\omega[(m^2-|\lambda|A^2)(2+\Omega_4)-\chi_4] \pm (m^2-|\lambda|A^2)^{3/2} \times [2\chi_4-(m^2-|\lambda|A^2)(1+2\Omega_4)]^{1/2}\sqrt{3}}{(m^2-|\lambda|A^2)[2k^2(2+\Omega_4)+3(m^2-|\lambda|A^2)]-2k^2\chi_4} \quad (39)$$

where

$$\Omega_4 = \frac{m^4}{|\lambda|A^2}\frac{1}{(|\lambda|A^2-m^2)}\left[\frac{2E}{K}-\left(1+\frac{2(|\lambda|A^2-m^2)}{(|\lambda|A^2-2m^2)}\frac{E^2}{K^2}\right)\right]$$

and (40)

$$\chi_4 = \left[2m^2\left(1-\frac{E}{K}\right)-|\lambda|A^2\right].$$

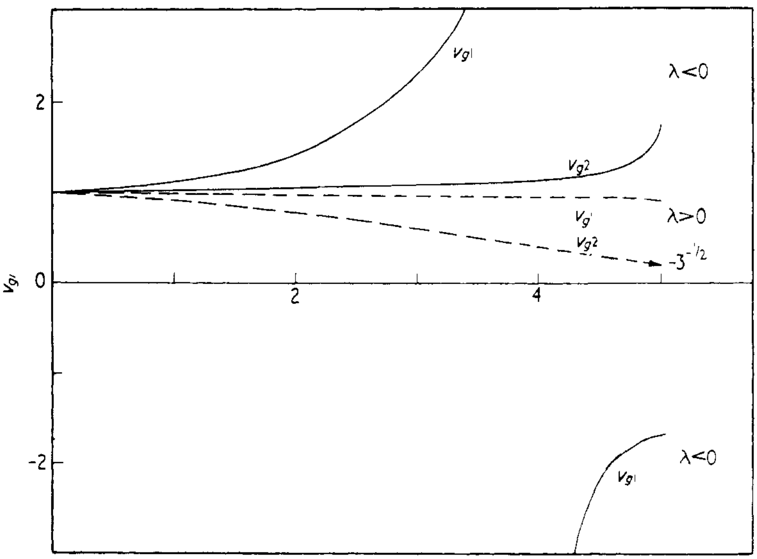

Figure 1. The velocities of propagation of the 'disturbance' waves in the case of a $\lambda\phi^4$ field with $m^2 = 0$ (for both $\lambda > 0$ and $\lambda < 0$).

As $|\lambda|A^2$ varies from $2m^2$ to ∞, χ_4 varies from $2m^2$ to $-|\lambda|A^2$ while Ω_4 varies from $-\infty$ to $2m^2$. Then the quantity $\{[2\chi_4-(m^2-|\lambda|A^2)(1+2\Omega_4)](m^2-|\lambda|A^2)\}^{1/2}$ will always be positive definite for $|\lambda|A^2 > 2m^2$. One can then proceed to the determination of characteristic forms and Riemann invariants as in § 3 and arrive at an equation similar to equation (26). The interesting point to notice is that the magnitude of the propagation velocities given by (39) is always greater than unity showing that the inhomogeneities propagate with tachyonic velocities. To see this more clearly we can take the $m^2 = 0$ limit of (39) which is

$$\tau_{m^2=0} = (2\omega k \pm |\lambda|A^2\surd 3)/(2k^2-3|\lambda|A^2), \qquad \omega = (k^2-|\lambda|A^2)^{1/2}. \tag{41}$$

One may verify that in (39) both the velocities are tachyonic. Considering

$$\tau_1 = (2\omega k + |\lambda|A^2\surd 3)/(2k^2-3|\lambda|A^2),$$

it varies from one to infinity as $|\lambda|A^2$ varies from zero to $\frac{2}{3}k^2$ and from $-\infty$ to $-\surd 3$ as $|\lambda|A^2$ varies from $\frac{2}{3}k^2$ to k^2 (which is the maximum allowed value of $|\lambda|A^2$). The other expression $\tau_2 = (2\omega k - |\lambda|A^2\surd 3)/(2k^2-3|\lambda|A^2)$ varies monotonically from one to $\surd 3$ as $|\lambda|A^2$ varies from zero to infinity. These facts are illustrated in figure 1.

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